Local existence of solutions for a p-Laplacian type equation with delay term and logarithmic nonlinearity

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Abstract

In this work, we deal with a p-Laplacian type equation with delay term and logarithmic nonlinearity. We consider the local existence of solutions by using the Faedo-Galerkin method under suitable conditions.

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1 Introduction

In this work, we study a p-Laplacian type equation with delay term and logarithmic nonlinearity

$$\begin{cases} |u_t|^{\rho} u_{tt} - \Delta u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)\,ds & \\ -\Delta u_{tt} + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) & \\ = ku\ln u & \text{in } \Omega\times(0,\infty), \\ u_t(x,t-\tau) = f_0(x,t-\tau) & x\in\Omega,\ t\in(0,\tau), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & x\in\Omega, \\ u(x,t)=0 & x\in\partial\Omega,\ t\geq 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n, $n\geq 1$, with smooth boundary $\partial\Omega$, $\rho>0$ and Δ denotes the Laplacian operator. $p>2$, μ_1, μ_2 and k are positive constants, $\tau>0$ represents time delay and the kernel of the memory term is g. The term $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is called p-Laplacian and $(u_0,\ u_1,\ f_0)$ are the functions of initial data to be specified later.

- **Problems with logarithmic nonlinearity:**

Logarithmic nonlinearity generally appears in super symmetric field theories and in cosmological inflation. From Quantum Field Theory, that such kind of $(u\ln|u|)$ logaritmic source term seems in nuclear physics, inflation cosmology, geophysics and optics (see [1, 2]).

Firstly, we begin with the studies of Birula and Mycielski [3, 4]. They investigated the equation with logarithmic term as follows

$$u_{tt} - u_{xx} + u - \varepsilon u\ln|u|^2 = 0. \tag{1.2}$$

They are the pioneer of these kind of problems which are the relativistic version of logarithmic quantum mechanics. In 1980, Cazenave and Haraux [5] looked into the equation as follows

$$u_{tt} - \Delta u = u\ln|u|^k. \tag{1.3}$$

They obtained the existence and uniqueness of the equation (1.3).

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- **Problems with time delay:**

Time delay appears in practical problems such as thermal, biological, chemical, physical and economic phenomena. Also, delay effects lead to instability. A small delay can destabilize a system which is uniformly asymptotically stable in the absence of delay unless control terms or additional conditions [6].

In 1986, Datko et al. [7] indicated that delay is a source of instability. In [8], Nicaise and Pignotti investigated the wave equation with delay term as follows

$$u_{tt} - \Delta u + \mu_1 u_t (x,t) + \mu_2 u_t (x, t-\tau) = 0. \tag{1.4}$$

Under the condition $0 < \mu_1 < \mu_2$, they proved the stability.

Recently, Andrade et al. [9] considered, this kind of problem without the memory term models elastoplastic flows, stability of the equation with p-Laplacian term as follows

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)\, ds - \Delta u_t + f(u) = 0. \tag{1.5}$$

Then, Araujo et al. [10] proved existence and uniqueness of weak solutions for the equation (1.5).

Messaoudi et al. [11], considered the viscoelastic plate equation with logarithmic term, they obtained existence and decay results. Mezouar et al. [12], looked into global existence and proved general stability estimate for the viscoelastic Kirchhoff equation with logarithmic term. Then, Pişkin and Yüksekkaya [13], studied local existence and blow up of solutions for the logarithmic nonlinear viscoelastic wave equation with delay. Also, some other authors concerned the theoretical and numerical analysis for some related problems (see [14, 15, 16, 17, 18]).

In the absence of the p-laplacian term ($\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$) and the logarithmic source term ($ku \ln u$), the problem (1.1) can be reduced

$$\left\{ \begin{array}{l} |u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\, ds \\ +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) \quad = 0. \end{array} \right. \tag{1.6}$$

Wu [19], considered local existence of solutions and proved decay result by suitable Lyapunov functionals for the equation (1.6).

There is no research, to our best knowledge, about local existence of solutions for p-Laplacian type equation with a varying material density ($|u_t|^{\rho}$), delay term ($\mu_2 u_t (x, t-\tau)$) and logarithmic source term ($ku \ln u$), therefore, our paper is generalization of the previous works.

The outline of this paper is as follows: Firstly, In Sect. 2, we give needed assumptions and lemmas which will be used throught this paper. Then, In Sect. 3, we get local existence of solutions by using Faedo-Galerkin method.

2 Preliminaries

In this part, we give needed lemmas and assumptions for our result. We use Lebesgue $L^p(\Omega)$ and Sobolev $H_0^1(\Omega)$ spaces with their usual scalar products and norms.

Lemma 2.1. [20, 21] (Sobolev-Poincare inequality) Let $2 \le q \le \frac{2n}{n-2}$, the inequality

$$\|u\|_q \le c_s \|\nabla u\|_2 \ \text{ for } u \in H_0^1(\Omega), \tag{2.1}$$

satisfies for some positive constant c_s.

Lemma 2.2. [22] Let $g \in C^1(R)$ and $\varphi \in H^1(0,T)$, such that

$$
\begin{aligned}
-2\int_0^t \int_\Omega g(t-s)\,\varphi\varphi_t dxds &= \frac{d}{dt}\left((g\circ\varphi)(t) - \int_0^t g(s)\,ds\,\|\varphi\|^2\right) \\
&\quad + g(t)\,\|\varphi\|^2 - (g'\circ\varphi)(t), \qquad (2.2)
\end{aligned}
$$

where

$$(g\circ\varphi)(t) = \int_0^t g(t-s)\int_\Omega |\varphi(s) - \varphi(t)|^2\,dxds.$$

Lemma 2.3. [23, 24] (Logarithmic Sobolev inequality) Assume that u is a function in $H_0^1(\Omega)$ and $a > 0$ is a number. We have,

$$\int_\Omega u^2 \ln|u|\,dx \le \frac{1}{2}\|u\|^2 \ln\|u\|^2 + \frac{a^2}{2\pi}\|\nabla u\|^2 - (1+\ln a)\,\|u\|^2. \qquad (2.3)$$

Lemma 2.4. [5] (Logarithmic Gronwall inequality) Suppose that $C > 0$, $\gamma \in L^1(0,T;R^+)$ and suppose that the function $w : [0,T] \to [1,\infty)$ satisfies

$$w(t) \le C\left(1 + \int_0^t \gamma(s)\,w(s)\ln(w(s))\,ds\right),\ \forall t \in [0,T]. \qquad (2.4)$$

Then,

$$w(t) \le C\exp\left(C\int_0^t \gamma(s)\,ds\right),\ \forall t \in [0,T]. \qquad (2.5)$$

Lemma 2.5. [11] Let $\varepsilon_0 \in (0,1)$. Then, there exists $d_{\varepsilon_0} > 0$ such that

$$s\,|\ln s| \le s^2 + d_{\varepsilon_0} s^{1-\varepsilon_0},\ \forall s > 0. \qquad (2.6)$$

Formulation of the results:

Firstly, we introduce, similar to [8], the new function

$$z(x,\kappa,t) = u_t(x, t-\tau\kappa),\ x \in \Omega,\ \kappa \in (0,1),$$

which gives us

$$\tau z_t(x,\kappa,t) + z_\kappa(x,\kappa,t) = 0 \text{ in } \Omega\times(0,1)\times(0,\infty).$$

Hence, problem (1.1) transforms into:

$$
\begin{cases}
|u_t|^\rho u_{tt} - \Delta u - \Delta_p u + \int_0^t g(t-s)\,\Delta u(s)\,ds & \\
-\Delta u_{tt} + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) & \\
= ku\ln u & \text{in } \Omega\times(0,\infty), \\
\tau z_t(x,\kappa,t) + z_\kappa(x,\kappa,t) = 0 & x\in\Omega,\ \kappa\in(0,1),\ t>0, \\
z(x,0,t) = u_t(x,t) & x\in\Omega,\ t>0, \\
z(x,\kappa,0) = f_0(x,-\tau\kappa) & x\in\Omega, \\
u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x\in\Omega, \\
u(x,t) = 0 & x\in\partial\Omega,\ t\ge 0.
\end{cases} \qquad (2.7)
$$

Now, we give the following assumptions to get our main result.

Suppose that ρ satisfies

$$0 < \rho \leq \frac{2}{n-2} \text{ if } n \geq 3 \text{ or } \rho > 0 \text{ if } n = 1, 2. \tag{2.8}$$

(A1) Relaxation function $g : R^{+} \to R^{+}$ is a bounded C^{1} function satisfies

$$1 - \int_0^{\infty} g(s)\, ds = l > 0, \tag{2.9}$$

and we suppose that, for $t \geq 0$, there exists a positive nonincreasing function ξ, such that

$$g'(t) \leq -\xi(t)\, g(t) \text{ and } \int_0^{\infty} \xi(s)\, ds = \infty. \tag{2.10}$$

(A2) k is the constant in (1.1) such that

$$1 < k < 2\pi l e^{3}. \tag{2.11}$$

(A3) We assume that

$$2 < p \leq \frac{2n-2}{n-2} \text{ if } n \geq 3 \text{ and } p > 2 \text{ if } n = 1,\ 2, \tag{2.12}$$

for $n \in N$. Hence,

$$H^{2}(\Omega) \cap H_0^{1}(\Omega) \hookrightarrow W_0^{1,2(p-1)}(\Omega) \hookrightarrow H_0^{1}(\Omega) \hookrightarrow L^{2}(\Omega).$$

Now, we define $E(t)$ energy functional related to system (2.7) by

$$\begin{aligned} E(t) &= \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^{t} g(s)\, ds\right)\|\nabla u\|^{2} + \frac{1}{2}\|\nabla u_t\|^{2} \\ &\quad + \frac{1}{p}\|\nabla u\|_p^{p} + \frac{1}{2}(g \circ \nabla u)(t) - \frac{k}{2}\int_{\Omega} |u|^{2}\ln|u|\, dx + \frac{k}{4}\|u\|^{2} \\ &\quad + \frac{\zeta}{2}\int_{\Omega}\int_0^{1} z^{2}(x,\kappa,t)\, d\kappa dx. \end{aligned} \tag{2.13}$$

Lemma 2.6. *$E(t)$ energy functional is a nonincreasing function on $[0,T]$, such that*

$$\begin{aligned} E'(t) &= -c_1\|u_t\|^{2} - c_2\int_{\Omega} z^{2}(x,1,s)\, dx + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^{2} \\ &\leq \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^{2} \\ &\leq 0,\ \forall t \geq 0. \end{aligned} \tag{2.14}$$

Proof. Multiplying the first equation in (2.7) by u_t and integrating over Ω and using integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{\rho+2}\left\|u_t\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left\|\nabla u\right\|^2+\frac{1}{p}\left\|\nabla u\right\|_p^p\right. \\
&\left.+\frac{1}{2}\left\|\nabla u_t\right\|^2-\frac{k}{2}\int_\Omega |u|^2\ln|u|\,dx+\frac{k}{4}\left\|u\right\|^2\right) \\
&+\mu_1\left\|u_t\right\|^2+\int_\Omega \mu_2 z\left(x,1,t\right)u_t dx \\
&-\int_0^t g\left(t-s\right)\int_\Omega \nabla u\left(s\right)\nabla u_t\left(t\right)dxds \\
=\;& 0. \qquad (2.15)
\end{aligned}$$

By applying Lemma 2.2 for the last term on the left hand side of (2.15), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{\rho+2}\left\|u_t\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left(1-\int_0^t g\left(s\right)ds\right)\left\|\nabla u\right\|^2+\frac{1}{p}\left\|\nabla u\right\|_p^p\right. \\
&\left.+\frac{1}{2}\left\|\nabla u_t\right\|^2+\frac{1}{2}\left(g\circ\nabla u\right)\left(t\right)-\frac{k}{2}\int_\Omega |u|^2\ln|u|\,dx+\frac{k}{4}\left\|u\right\|^2\right) \\
&+\mu_1\left\|u_t\right\|^2+\int_\Omega \mu_2 z\left(x,1,t\right)u_t dx+\frac{1}{2}g\left(t\right)\left\|\nabla u\right\|^2-\frac{1}{2}\left(g'\circ\nabla u\right)\left(t\right) \\
=\;& 0. \qquad (2.16)
\end{aligned}$$

We multiply the second equation in (2.7) by $\frac{\zeta}{\tau}z$ and integrate over $\Omega\times(0,1)$, $\zeta>0$, we have

$$\frac{\zeta}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2\left(x,\kappa,t\right)d\kappa dx+\frac{\zeta}{\tau}\int_\Omega\int_0^1 z\left(x,\kappa,t\right)z_\kappa\left(x,\kappa,t\right)d\kappa dx=0. \qquad (2.17)$$

We note that

$$\begin{aligned}
&-\frac{\zeta}{\tau}\int_\Omega\int_0^1 z\left(x,\kappa,t\right)z_\kappa\left(x,\kappa,t\right)d\kappa dx \\
=\;& -\frac{\zeta}{2\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\kappa}z^2\left(x,\kappa,t\right)d\kappa dx \\
=\;& \frac{\zeta}{2\tau}\int_\Omega\left(z^2\left(x,0,t\right)-z^2\left(x,1,t\right)\right)dx \\
=\;& \frac{\zeta}{2\tau}\left(\int_\Omega u_t^2 dx-\int_\Omega z^2\left(x,1,t\right)dx\right). \qquad (2.18)
\end{aligned}$$

Combining (2.16) and (2.17) and taking into consideration (2.18), we get

$$\begin{aligned}
&E'\left(t\right)=-\left(\mu_1-\frac{\zeta}{2\tau}\right)\left\|u_t\right\|^2 \\
&-\frac{\zeta}{2\tau}\int_\Omega z^2\left(x,1,t\right)dx-\mu_2\int_\Omega z\left(x,1,t\right)u_t dx \\
&-\frac{1}{2}g\left(t\right)\left\|\nabla u\left(t\right)\right\|^2+\frac{1}{2}\left(g'\circ\nabla u\right)\left(t\right). \qquad (2.19)
\end{aligned}$$

Using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ on the third term of the right-hand side of (2.19), we arrive

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\|u_t\|^2 - \left(\frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\int_\Omega z^2(x,1,t)\,dx \\ &\quad -\frac{1}{2}g(t)\|\nabla u(t)\|^2 + \frac{1}{2}(g' \circ \nabla u)(t) \\ &\leq \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2 \leq 0,\ \forall t \geq 0. \end{aligned} \tag{2.20}$$

Choosing ζ such that

$$\tau\mu_2 < \zeta < \tau(2\mu_1 - \mu_2), \tag{2.21}$$

which satisfies

$$c_1 = \mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2} > 0 \text{ and } c_2 = \frac{\zeta}{2\tau} - \frac{\mu_2}{2} > 0,$$

due to $\mu_1 > \mu_2$. Q.E.D.

3 Local existence

In this part, we get local existence of solutions for (2.7) by utilizing Faedo-Galerkin method.

Theorem 3.1. Let $\mu_2 < \mu_1$, (2.8) and (A1)-(A3) hold. Suppose that $u_0,\ u_1 \in W_0^{1,p}$ and $f_0 \in L^2(\Omega \times (0,1))$. Then, there exists a unique solution (u,z) of (2.7) satisfies

$$\begin{aligned} u,\ u_t &\in C\left([0,T); W_0^{1,p}(\Omega)\right), \\ z &\in C([0,T)); L^2(\Omega \times (0,1)), \end{aligned}$$

for $T > 0$.

Proof. Let (w^k), $k \in N$, be a basis in $H_0^1(\Omega)$ and W_k be the space generated by $w_1, \cdots, w^k$, $k = 1$, 2, 3,$\cdots$. Now, we define for $1 \leq j \leq k$, the sequence $\varphi^j(x,\kappa)$ as follows:

$$\varphi^j(x,0) = w^j(x).$$

Hence, we can extend $\varphi^j(x,0)$ by $\varphi^j(x,\kappa)$ over $L^2(\Omega \times [0,1])$ and denote V_k to be the space generated by $\varphi_1, \cdots, \varphi^k$, $k = 1$, 2, 3,$\cdots$. Let us consider

$$u^k(t) = \sum_{j=1}^{k} c^{jk}(t)\,w^j(x),$$

and

$$z^k(t) = \sum_{j=1}^{k} r^{jk}(t)\,\varphi^j(x,\kappa),$$

where (u^k,z^k) are the solutions, corresponding to (2.7), of approximate problem as follows:

$$\begin{aligned}
&\int_\Omega \left|u_t^k\right|^\rho u_{tt}^k w^j dx + \int_\Omega \nabla u^k \cdot \nabla w^j dx \\
&-\int_0^t g(t-\tau) \int_\Omega \nabla u^k(\tau) \cdot \nabla w^j dx d\tau \\
&+\int_\Omega \nabla u_{tt}^k \cdot \nabla w^j dx + \int_\Omega \left|\nabla u^k\right|^{p-2} \left|\nabla u^k\right| \cdot \nabla w^j dx \\
&+\int_\Omega \left(\mu_1 u_t^k(x,t) + \mu_2 z^k(x,1,t)\right) w^j dx \\
= \; & k\int_\Omega \left(u^k \ln\left|u^k\right|\right) w^j dx. \qquad (3.1)
\end{aligned}$$

$$u^k(0) = u_0^k \to u_0 \text{ in } H_0^1(\Omega),\; u_t^k(0) = u_1^k \to u_1 \text{ in } H_0^1(\Omega), \qquad (3.2)$$

and

$$\int_\Omega \left(\tau z_t^k(x,\kappa,t) + z_\kappa^k(x,\kappa,t)\right) \varphi^j dx = 0, \qquad (3.3)$$

$$z^k(0) = z_0^k \to f_0 \text{ in } L^2(\Omega\times(0,1)), \qquad (3.4)$$

here $i = 1,2,\cdots,k$. Utilizing (2.8) and Hölder inequality, the nonlinear term $\int_\Omega \left|u_t^k\right|^\rho u_{tt}^k(t) w^j dx$ makes sense in (3.1). Then, by standard methods in ordinary differential equations, we conclude that the existence of solutions to (3.1)-(3.4) on some interval $\left[0,t^k\right)$, $0<t^k<T$ for some arbitrary $T>0$. Now, we get a priori estimate for the solution (3.1)-(3.4), hence, the solution can be extended to the whole interval $[0,T)$. Q.E.D.

First estimate:

The sequences u_0^k, u_1^k and z_0^k are converge, thus, by using Lemma 2.6 we find a positive constant C_1 independent of k, such that

$$\begin{aligned}
&E^k(t) - E^k(0) \\
\le\; & -c_1\int_0^t \left\|u_t^k\right\|^2 ds - c_2\int_0^t\int_\Omega \left|z^k(x,1,s)\right|^2 dx ds \\
&-\frac{1}{2}\int_0^t g(s)\left\|\nabla u^k(s)\right\|^2 ds + \frac{1}{2}\int_0^t \left(g'\circ\nabla u^k\right)(s) ds \\
\le\; & -c_1\int_0^t \left\|u_t^k\right\|^2 ds - c_2\int_0^t\int_\Omega \left|z^k(x,1,s)\right|^2 dx ds, \qquad (3.5)
\end{aligned}$$

g is a positive nonincreasing function, thus, we obtain

$$\begin{aligned}
&E^k(t) + c_1\int_0^t \left\|u_t^k\right\|^2 ds + c_2\int_0^t\int_\Omega \left|z^k(x,1,s)\right|^2 dx ds \\
\le\; & E^k(0) \le C_1, \qquad (3.6)
\end{aligned}$$

where

$$\begin{aligned} E^k(t) &= \frac{1}{\rho+2}\left\|u_t^k\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\left\|\nabla u^k\right\|^2+\frac{1}{2}\left\|\nabla u_t^k\right\|^2 \\ &\quad +\frac{1}{p}\left\|\nabla u^k\right\|_p^p+\frac{1}{2}\left(g\circ\nabla u^k\right)(t)-\frac{k}{2}\int_\Omega\left|u^k\right|^2\ln\left|u^k\right|dx \\ &\quad +\frac{k}{4}\left\|u^k\right\|^2+\frac{\zeta}{2}\int_\Omega\int_0^1\left|z^k(x,\kappa,t)\right|^2 d\kappa dx. \end{aligned} \tag{3.7}$$

Utilizing the Logarithmic Sobolev inequality, (3.6) yields

$$\begin{aligned} &\left\|u_t^k\right\|_{\rho+2}^{\rho+2}+\left(l-\frac{ka^2}{2\pi}\right)\left\|\nabla u^k\right\|^2+\left[\frac{k}{2}+k(1+\ln a)\right]\left\|u^k\right\|^2+\left\|\nabla u_t^k\right\|^2 \\ &+\frac{1}{p}\left\|\nabla u^k\right\|_p^p+\left(g\circ\nabla u^k\right)(t)+\int_\Omega\int_0^1\left|z^k(x,\kappa,t)\right|^2 d\kappa dx \\ &+\int_0^t\left\|u_t^k\right\|^2 ds+\int_0^t\int_\Omega\left|z^k(x,1,s)\right|^2 dxds \\ \leq\ & C_2+\left\|u^k\right\|^2\ln\left\|u^k\right\|^2, \end{aligned} \tag{3.8}$$

where C_2 is a positive constant.

Choosing

$$e^{(-3/2)}<a<\sqrt{\frac{2\pi l}{k}}, \tag{3.9}$$

will make

$$l-\frac{ka^2}{2\pi}>0$$

and

$$\frac{k}{2}+k(1+\ln a)>0.$$

Thanks to (A2) this selection is possible. Hence, we obtain

$$\begin{aligned} &\left\|u_t^k\right\|_{\rho+2}^{\rho+2}+\left\|\nabla u^k\right\|^2+\left\|u^k\right\|^2+\left\|\nabla u_t^k\right\|^2 \\ &+\left\|\nabla u^k\right\|_p^p+\left(g\circ\nabla u^k\right)(t)+\int_\Omega\int_0^1\left|z^k(x,\kappa,t)\right|^2 d\kappa dx \\ &+\int_0^t\left\|u_t^k\right\|^2 ds+\int_0^t\int_\Omega\left|z^k(x,1,s)\right|^2 dxds \\ \leq\ & c\left(1+\left\|u^k\right\|^2\ln\left\|u^k\right\|^2\right). \end{aligned} \tag{3.10}$$

Let us note that

$$u^k(t)=u^k(0)+\int_0^t u_s^k(s)\,ds. \tag{3.11}$$

Then, utilizing Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}\left\|u^k\right\|^2 &\leq \left\|u^k(0)\right\|^2 + 2\left\|\int_0^t u_s^k(s)\,ds\right\|^2 \\ &\leq \left\|u^k(0)\right\|^2 + 2T\int_0^t \left\|u_s^k(s)\right\|^2 ds. \end{aligned} \tag{3.12}$$

Therefore, (3.10) gives

$$\left\|u^k\right\|^2 \leq C\left(1+\int_0^t \left\|u^k\right\|^2 \ln\left\|u^k\right\|^2 ds\right), \tag{3.13}$$

where $C = \max\left\{2Tc, 2\left\|u^k(0)\right\|^2\right\}$. By using the Logarithmic Gronwall inequality to (3.13), we get

$$\left\|u^k\right\|^2 \leq Ce^{CT}. \tag{3.14}$$

Thus, by (3.10), we have the first estimate:

$$\begin{aligned} & \left\|u_t^k\right\|_{\rho+2}^{\rho+2} + \left\|\nabla u^k\right\|^2 + \left\|\nabla u_t^k\right\|^2 + \left\|u^k\right\|^2 + \left\|\nabla u^k\right\|_p^p \\ & + \left(g\circ\nabla u^k\right)(t) + \int_\Omega\int_0^1 \left|z^k(x,\kappa,t)\right|^2 d\kappa dx \\ & + \int_0^t \left\|u_t^k\right\|^2 ds + \int_0^t\int_\Omega \left|z^k(x,1,s)\right|^2 dxds \\ \leq\ & c\left(1+Ce^{CT}\ln\left(Ce^{CT}\right)\right) = A_1. \end{aligned} \tag{3.15}$$

Second estimate:
We replace w^j by $-\Delta w^j$ in (3.1), multiply by c_t^{jk} and sum up over j from 1 to k, to have

$$\begin{aligned} & \int_\Omega \left|u_t^k(t)\right|^\rho u_{tt}^k(t)\left(-\Delta u_t^k\right)dx + \int_\Omega \Delta u^k \Delta u_t^k dx + \int_\Omega \Delta_p u^k \Delta u_t^k dx \\ & -\int_0^t g(t-s)\int_\Omega \Delta u^k \Delta u_t^k dxds + \int_\Omega \Delta u_{tt}^k \Delta u_t^k dx \\ & -\mu_1\int_\Omega u_t^k \Delta u_t^k dx - \mu_2\int_\Omega \Delta u_t^k z^k(x,1,t)\,dx \\ =\ & -k\int_\Omega \Delta u_t^k u^k \ln\left|u^k\right| dx. \end{aligned} \tag{3.16}$$

Using Lemma 2.2, we get

$$\begin{aligned} & -\int_\Omega \left|u_t^k(t)\right|^\rho u_{tt}^k(t)\,\Delta u_t^k dx + \frac{1}{2}g(t)\left\|\Delta u^k\right\|^2 - \frac{1}{2}\left(g'\circ\Delta u^k\right) \\ & +\frac{1}{2}\frac{d}{dt}\left[\left(1-\int_0^t g(s)\,ds\right)\left\|\Delta u^k\right\|^2 + \left\|\Delta u_t^k\right\|^2 + \left(g\circ\Delta u^k\right)\right] \\ & +\int_\Omega \Delta_p u^k \Delta u_t^k dx + \mu_1\int_\Omega \left|\nabla u_t^k\right|^2 dx + \mu_2\int_\Omega \nabla u_t^k \nabla z^k(x,1,t)\,dx \\ =\ & -k\int_\Omega \Delta u_t^k u^k \ln\left|u^k\right| dx. \end{aligned} \tag{3.17}$$

Utilizing the Green's formula, we get

$$\begin{aligned} & -\int_\Omega \left|u_t^k(t)\right|^\rho u_{tt}^k \Delta u_t^k dx \\ = & \frac{d}{dt}\left[\int_\Omega \left|u_t^k(t)\right|^\rho \left|\nabla u_t^k\right|^2 dx\right] - (\rho+1)\int_\Omega \left|u_t^k(t)\right|^\rho \nabla u_{tt}^k \nabla u_t^k dx. \end{aligned} \tag{3.18}$$

As a result, equation (3.17) becomes

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left[2\int_\Omega \left|u_t^k(t)\right|^\rho \left|\nabla u_t^k\right|^2 dx + \left(1-\int_0^t g(s)\,ds\right)\left\|\Delta u^k\right\|^2 + \left\|\Delta u_t^k\right\|^2 + \left(g\circ \Delta u^k\right)\right] \\ & -(\rho+1)\int_\Omega \left|u_t^k(t)\right|^\rho \nabla u_{tt}^k \nabla u_t^k dx + \frac{1}{2}g(t)\left\|\Delta u^k\right\|^2 - \frac{1}{2}\left(g'\circ \Delta u^k\right) \\ & +\mu_1\int_\Omega \left|\nabla u_t^k\right|^2 dx + \mu_2\int_\Omega \nabla u_t^k \nabla z^k(x,1,t)\,dx + \int_\Omega \Delta_p u^k \Delta u_t^k dx \\ = & -k\int_\Omega \Delta u_t^k u^k \ln\left|u^k\right| dx. \end{aligned} \tag{3.19}$$

For estimating the last term on the left-hand side of (3.19), we note that

$$\int_\Omega \Delta_p u^k \Delta u_t^k dx = \frac{d}{dt}\int_\Omega \Delta_p u^k \Delta u^k dx - J_1, \tag{3.20}$$

where

$$J_1 = \int_\Omega \left\{(p-2)\left|\nabla u^k\right|^{p-4}\left(\nabla u^k\cdot \nabla u_t^k\right)\nabla u^k + \left|\nabla u^k\right|^{p-2}\nabla u_t^k\right\}\cdot \nabla \Delta u^k dx. \tag{3.21}$$

Hence, (3.19) becomes

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left[2\int_\Omega \left|u_t^k(t)\right|^\rho \left|\nabla u_t^k\right|^2 dx + \left(1-\int_0^t g(s)\,ds\right)\left\|\Delta u^k\right\|^2\right. \\ & \left. +\left\|\Delta u_t^k\right\|^2 + \left(g\circ\Delta u^k\right) + \int_\Omega \Delta_p u^k \Delta u^k dx\right] \\ & -(\rho+1)\int_\Omega \left|u_t^k(t)\right|^\rho \nabla u_{tt}^k \nabla u_t^k dx + \frac{1}{2}g(t)\left\|\Delta u^k\right\|^2 \\ & -\frac{1}{2}\left(g'\circ\Delta u^k\right) + \mu_1\int_\Omega \left|\nabla u_t^k\right|^2 dx + \mu_2\int_\Omega \nabla u_t^k \nabla z^k(x,1,t)\,dx \\ = & J_1 - k\int_\Omega \Delta u_t^k u^k \ln\left|u^k\right| dx. \end{aligned} \tag{3.22}$$

By using the estimate (3.15) and $\frac{p-2}{2(p-1)} + \frac{1}{2(p-1)} + \frac{1}{2} = 1$, we obtain

$$\begin{aligned} |J_1| & \leq (p-1)\int_\Omega \left|\nabla u^k(t)\right|^{p-2}\left|\nabla u_t^k(t)\right|\left|\nabla\Delta u^k(t)\right| dx \\ & \leq (p-1)\left\|\nabla u^k(t)\right\|_{2(p-1)}^{p-2}\left\|\nabla u_t^k(t)\right\|_{2(p-1)}\left\|\nabla\Delta u^k(t)\right\|_2 \\ & \leq C\left\|\nabla u_t^k(t)\right\|_{2(p-1)}\left\|\nabla\Delta u^k(t)\right\|_2. \end{aligned} \tag{3.23}$$

Because $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow W_0^{1,2(p-1)}(\Omega)$ and u_t^k is regular, we get

$$\left\| \nabla u_t^k(t) \right\|_{2(p-1)}^2 \leq \mu_2 \left\| \Delta u_t^k(t) \right\|_2^2, \tag{3.24}$$

here $\mu_2 > 0$ is embedding constant. Hence,

$$|J_1| \leq \frac{1}{4} \left\| \Delta u_t^k(t) \right\|_2^2 + C \left\| \nabla \Delta u^k(t) \right\|_2^2. \tag{3.25}$$

We note that

$$\begin{aligned} \left| \int_\Omega \Delta_p u^k \Delta u^k dx \right| &\leq \int_\Omega \left| \nabla u^k(t) \right|^{p-1} \left| \nabla \Delta u^k(t) \right| dx \\ &\leq \left\| \nabla u^k(t) \right\|_{2(p-1)}^{p-1} \left\| \nabla \Delta u^k(t) \right\|_2 \\ &\leq C + \left\| \nabla \Delta u^k(t) \right\|_2^2, \end{aligned} \tag{3.26}$$

and then

$$C + \left\| \nabla \Delta u^k(t) \right\|_2^2 + \int_\Omega \Delta_p u^k \Delta u^k dx \geq 0. \tag{3.27}$$

Hence, there exists $C_0 > 0$, (3.22) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[2\int_\Omega \left| u_t^k(t) \right|^\rho \left| \nabla u_t^k \right|^2 dx + \left(1 - \int_0^t g(s)\,ds \right) \left\| \Delta u^k \right\|^2 \right. \\ &\left. + \left\| \Delta u_t^k \right\|^2 + \left(g \circ \Delta u^k \right) + \int_\Omega \Delta_p u^k \Delta u^k dx \right] \\ &- (\rho + 1) \int_\Omega \left| u_t^k(t) \right|^\rho \nabla u_{tt}^k \nabla u_t^k dx + \frac{1}{2} g(t) \left\| \Delta u^k \right\|^2 \\ &- \frac{1}{2} \left(g' \circ \Delta u^k \right) + \mu_1 \int_\Omega \left| \nabla u_t^k \right|^2 dx + \mu_2 \int_\Omega \nabla u_t^k \nabla z^k(x,1,t)\,dx \\ \leq\; & C_0 + C_0 \left\| \nabla \Delta u^k \right\|_2^2 + \int_\Omega \Delta_p u^k \Delta u^k dx \\ &+ \frac{1}{4} \left\| \Delta u_t^k(t) \right\|_2^2 - k \int_\Omega \Delta u_t^k u^k \ln \left| u^k \right| dx. \end{aligned} \tag{3.28}$$

Estimating the term on the right-hand side of (3.19), applying Lemma 2.5 with $\varepsilon_0 = (1/2)$ and from embedding, Young's and Cauchy-Schwartz's inequalities, we get:

$$\begin{aligned} \left| k \int_\Omega \Delta u_t^k u^k \ln \left| u^k \right| dx \right| &\leq k \int_\Omega \left| \Delta u_t^k \right| \left(\left| u^k \right|^2 + d_{\varepsilon_0} \sqrt{\left| u^k \right|} \right) dx \\ &\leq k \left(\eta \int_\Omega \left| \Delta u_t^k \right|^2 dx + \frac{1}{4\eta} \int_\Omega \left(\left| u^k \right|^2 + d_{\varepsilon_0} \sqrt{\left| u^k \right|} \right)^2 dx \right) \\ &\leq k\eta \int_\Omega \left| \Delta u_t^k \right|^2 dx + \frac{c}{4\eta} \left(\int_\Omega \left| u^k \right|^4 dx + \int_\Omega \left| u^k \right| dx \right) \\ &\leq k\eta \left\| \Delta u_t^k \right\|^2 + \frac{c}{4\eta} \left(\left\| \nabla u^k \right\|^4 + \left\| u^k \right\| \right), \; \eta > 0. \end{aligned} \tag{3.29}$$

Taking into account (3.20)-(3.28) and substituting (3.29) into (3.19), we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left[2\int_\Omega \left|u_t^k\right|^\rho \left|\nabla u_t^k\right|^2 dx + \left(1-\int_0^t g\left(s\right)ds\right)\left\|\Delta u^k\right\|^2 \right. \\ & \left. +\left\|\Delta u_t^k\right\|^2 + \left(g\circ \Delta u^k\right) + \int_\Omega \Delta_p u^k \Delta u^k dx\right] \\ & -\left(\rho+1\right)\int_\Omega \left|u_t^k\left(t\right)\right|^\rho \nabla u_{tt}^k \nabla u_t^k dx \\ & +\frac{1}{2}g\left(t\right)\left\|\Delta u^k\right\|^2 - \frac{1}{2}\left(g'\circ \Delta u^k\right) \\ & +\mu_1 \int_\Omega \left|\nabla u_t^k\right|^2 dx + \mu_2 \int_\Omega \nabla u_t^k \nabla z^k\left(x,1,t\right)dx \\ \leq \;& C_0 + C_0\left\|\nabla \Delta u^k\right\|_2^2 + \int_\Omega \Delta_p u^k \Delta u^k dx + \frac{1}{4}\left\|\Delta u_t^k\left(t\right)\right\|_2^2 \\ & +k\eta \left\|\Delta u_t^k\right\|^2 + \frac{c}{4\eta}\left(\left\|\nabla u^k\right\|^4 + \left\|u^k\right\|\right). \end{aligned} \tag{3.30}$$

Replacing φ^j by $-\Delta\varphi^j$ in (3.3), multiply by d^{jk} and sum up over j from 1 to k, we get

$$\tau\int_\Omega \nabla z_t^k \nabla z^k dx + \int_\Omega \nabla z_\rho^k \nabla z^k dx = 0. \tag{3.31}$$

Then, we obtain

$$\frac{\tau}{2}\frac{d}{dt}\left\|\nabla z^k\right\|^2 + \frac{1}{2}\frac{d}{d\rho}\left\|\nabla z^k\right\|^2 = 0. \tag{3.32}$$

Integrating over $(0,1)$, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_0^1 \tau\left\|\nabla z^k\left(x,\rho,t\right)\right\|^2 d\rho + \frac{1}{2}\left\|\nabla z^k\left(x,1,t\right)\right\|^2 \\ = \;& \frac{1}{2}\left\|\nabla u_t^k\right\|^2. \end{aligned} \tag{3.33}$$

Combining (3.30) and (3.33), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[2\int_\Omega \left|u_t^k(t)\right|^\rho \left|\nabla u_t^k\right|^2 dx + \left(1-\int_0^t g(s)\,ds\right)\left\|\Delta u^k\right\|^2 + \left\|\Delta u_t^k\right\|^2\right.\\
&\left. + \left(g\circ \Delta u^k\right) + \int_\Omega \Delta_p u^k \Delta u^k dx + \tau\int_0^1 \left\|\nabla z^k(x,\rho,t)\right\|^2 d\rho\right]\\
&+\frac{1}{2}g(t)\left\|\Delta u^k\right\|^2 - \frac{1}{2}\left(g'\circ\Delta u^k\right)\\
&+\mu_1\int_\Omega \left|\nabla u_t^k\right|^2 dx + \frac{1}{2}\left\|\nabla z^k(x,1,t)\right\|^2\\
\le\ & (\rho+1)\int_\Omega \left|u_t^k(t)\right|^\rho \nabla u_{tt}^k \nabla u_t^k dx - \mu_2\int_\Omega \nabla u_t^k \nabla z^k(x,1,t)\,dx\\
&+C_0 + C_0\left\|\nabla\Delta u^k\right\|_2^2 + \int_\Omega \Delta_p u^k\Delta u^k dx + \frac{1}{2}\left\|\nabla u_t^k\right\|^2\\
&+\frac{1}{4}\left\|\Delta u_t^k\right\|^2 + k\eta\left\|\Delta u_t^k\right\|^2 + \frac{c}{4\eta}\left(\left\|\nabla u^k\right\|^4 + \left\|u^k\right\|\right). \qquad (3.34)
\end{aligned}$$

Utilizing Young's inequality and from the first estimate (3.15), we obtain

$$\begin{aligned}
\int_\Omega \left|u_t^k\right|^\rho \nabla u_{tt}^k \nabla u_t^k dx &\le A_1^{(\rho/(\rho+2))+(1/2)}\left\|\nabla u_{tt}^k(t)\right\|_2\\
&\le \eta\left\|\nabla u_{tt}^k\right\|^2 + \frac{A_1^{(2\rho/(\rho+2))+1}}{4\eta}. \qquad (3.35)
\end{aligned}$$

From Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}
&\left|\int_\Omega \nabla u_t^k \nabla z^k(x,1,t)\,dx\right|\\
\le\ & \frac{\varepsilon}{2}\int_\Omega \left|\nabla u_t^k\right|^2 dx + \frac{1}{2\varepsilon}\int_\Omega \left|\nabla z^k(x,1,t)\right|^2 dx. \qquad (3.36)
\end{aligned}$$

Combining (3.34), (3.35) and (3.36), using (A1), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[2\int_\Omega \left|u_t^k(t)\right|^\rho \left|\nabla u_t^k\right|^2 dx + \left(1-\int_0^t g(s)\,ds\right)\left\|\Delta u^k\right\|^2 + \left\|\Delta u_t^k\right\|^2\right.\\
&\left. + \left(g\circ \Delta u^k\right) + \int_\Omega \Delta_p u^k \Delta u^k dx + \tau\int_0^1 \left\|\nabla z^k(x,\kappa,t)\right\|^2 d\rho\right]\\
&+\mu_1\int_\Omega \left|\nabla u_t^k\right|^2 dx + c\left\|\nabla z^k(x,1,t)\right\|^2\\
\le\ & (\rho+1)\eta\left\|\nabla u_{tt}^k\right\|^2 + c'\left\|\nabla u_t^k\right\|^2 + C'_\eta A_1 + \frac{1}{2}\left\|\nabla u_t^k\right\|^2\\
&+\frac{1}{4}\left\|\Delta u_t^k\right\|^2 + C_0 + C_0\left\|\nabla\Delta u^k\right\|_2^2 + \int_\Omega \Delta_p u^k\Delta u^k dx\\
&+k\eta\left\|\Delta u_t^k\right\|^2 + \frac{c}{4\eta}\left(\left\|\nabla u^k\right\|^4 + \left\|u^k\right\|\right). \qquad (3.37)
\end{aligned}$$

From (3.15) and by integrating over (0,t), (3.37) yields

$$\begin{aligned}
& 2\int_\Omega \left|u_t^k(t)\right|^\rho \left|\nabla u_t^k\right|^2 dx + \left(1-\int_0^t g(s)\,ds\right)\left\|\Delta u^k\right\|^2 + \left\|\Delta u_t^k\right\|^2 + \left(g\circ \Delta u^k\right) \\
& + \int_\Omega \Delta_p u^k \Delta u^k dx + \tau \int_0^1 \left\|\nabla z^k(x,\kappa,t)\right\|^2 d\rho \\
& + c_* \int_0^t \left\|\nabla u_t^k\right\|^2 ds + c\int_0^t \left\|\nabla z^k(x,1,t)\right\|^2 ds \\
\leq\ & \int_0^t (\rho+1)\,\eta \left\|\nabla u_{tt}^k\right\|^2 ds + \left(c'_\eta A_1 + C_0\right)T + \int_0^t \int_\Omega \Delta_p u^k \Delta u^k dx ds \\
& + \frac{1}{4}\int_0^t \left\|\Delta u_t^k\right\|^2 ds + \int_0^t \left\|\nabla \Delta u^k(t)\right\|^2 ds \\
& + k\eta \int_0^t \left\|\Delta u_t^k\right\|^2 ds + \int_0^t \frac{c}{4\eta}\left(\left\|\nabla u^k\right\|^4 + \left\|u^k\right\|\right) ds. \qquad (3.38)
\end{aligned}$$

Using Gronwall lemma, we get

$$\begin{aligned}
& \left\|\Delta u^k\right\|^2 + \left(g\circ\Delta u^k\right) + \int_0^1 \left\|\nabla z^k(x,\kappa,t)\right\|^2 d\rho + \left\|\nabla u_t^k\right\|^2 \\
& + \left\|\nabla\Delta u^k\right\|^2 + \int_0^t \left\|\Delta u_t^k\right\|^2 ds + \int_0^t \left\|\nabla u_{tt}^k\right\|^2 ds \\
\leq\ & M_2. \qquad (3.39)
\end{aligned}$$

We see that the estimates (3.15) and (3.39) imply that there exists a subsequence (u^m) of (u^k) and functions u, z, hence

$$u^m \to u \text{ weakly star in } L^\infty\left(0;T,W_0^{1,p}(\Omega)\right), \qquad (3.40)$$

$$u_t^m \to u_t \text{ weakly star in } L^\infty\left(0,T;L^{\rho+2}(\Omega)\right)\cap L^\infty\left(0,T,H_0^2(\Omega)\right), \qquad (3.41)$$

$$u_{tt}^m \to u_{tt} \text{ weakly star in } L^\infty\left(0,T;H_0^1(\Omega)\right), \qquad (3.42)$$

$$z^m \to z \text{ weakly star in } L^\infty\left(0,T;L^2(\Omega\times(0,1))\right), \qquad (3.43)$$

$$u^m \to u \text{ weakly in } L^2\left(0,T;H_0^1(\Omega)\right), \qquad (3.44)$$

$$u_t^m \to u_t \text{ weakly in } L^2\left(0,T;L^{\rho+2}(\Omega)\right)\cap L^2\left(0,T,H_0^2(\Omega)\right), \qquad (3.45)$$

$$u_{tt}^m \to u_{tt} \text{ weakly in } L^2\left(0,T;H_0^1(\Omega)\right). \qquad (3.46)$$

Analysis of the non-linear terms:

First Term ($u^k \ln\left|u^k\right|$): using (3.39), we get (u^k) is bounded in $L^\infty\left(0,T;H_0^2(\Omega)\right)$ which implies, utilizing the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$, the boundness of (u^k) in $L^2(\Omega\times(0,T))$. In a similar way, (u_t^k) is bounded in $L^2(\Omega\times(0,T))$. Therefore, by Aubin-Lions theorem [25], we find a subsequence such that

$$u^m \to u \text{ strongly in } L^2(\Omega\times(0,T)), \qquad (3.47)$$

hence

$$u^m \to u \text{ a.e. in } \Omega \times (0,T). \tag{3.48}$$

The maps $s \to ks \ln |s|$ is continuous, thus, we obtain the following convergence:

$$ku^m \ln |u^m| \to ku \ln |u| \text{ a.e. in } \Omega \times (0,T). \tag{3.49}$$

Using the embedding of $H_0^1(\Omega)$ in $L^\infty(\Omega)$, it is easy to see that $k(u^m \ln |u^m|)$ is bounded in $L^\infty(\Omega \times (0,T))$. Then, taking consideration the Lebesgue bounded convergence theorem, we have

$$ku^m \ln |u^m| \to ku \ln |u| \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right). \tag{3.50}$$

Second Term ($\left|u_t^k\right|^\rho u_t^k$):

From the first estimate in (3.15) and Lemma 2.1, we conclude

$$\begin{aligned} \left\| \left|u_t^k\right|^\rho u_t^k \right\|_{L^2(0,T,L^2(\Omega))} &= \int_0^T \left\|u_t^k\right\|_{2(\rho+1)}^{2(\rho+1)} dt \le C_s^{\,2(\rho+1)} \int_0^T \left\|\nabla u_t^k\right\|^{2(\rho+1)} dt \\ &\le C_s^{\,2(\rho+1)} A_1^{2(\rho+1)} T. \end{aligned} \tag{3.51}$$

Now, utilizing Aubin-Lions theorem, (see Lions [25]), there exists a subsequence of $\{u^m\}$, still denoted by $\{u^m\}$ therefore,

$$u_t^m \to u_t \text{ strongly in } L^2\left(0,T,L^2(\Omega)\right), \tag{3.52}$$

and

$$u_t^m \to u_t \text{ almost everywhere in } \Omega \times (0,T). \tag{3.53}$$

Therefore,

$$|u_t^m|^\rho u_t^m \to |u_t|^\rho u_t \text{ almost everywhere in } \Omega \times (0,T). \tag{3.54}$$

Hence, by using (3.47), (3.54) and using Lions Lemma, we obtain

$$|u_t^m|^\rho u_t^m \to |u_t|^\rho u_t \text{ weakly in } L^2\left(0,T,L^2(\Omega)\right), \tag{3.55}$$

and

$$z^m \to z \text{ strongly in } L^2\left(0,T,L^2(\Omega)\right), \tag{3.56}$$

which implies $z^m \to z$ almost everywhere in $\Omega \times (0,T)$.

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